28 The barber paradox

In a village there lives a barber who shaves all and only the people who do not shave themselves. So who shaves the barber? If he shaves himself, he does not; if he does not shave himself, he does.

On the face of it, the puzzle at the centre of the barber paradox may not seem too hard to fathom. A scenario that at first glance looks plausible quickly collapses into contradiction. The innocent-looking job description (a man 'who shaves all and only the people who do not shave themselves') is in fact logically impossible, since the barber cannot, without contradicting the description of himself, belong either to the group who shave themselves or to the group who do not. A man fitting the description of the barber cannot (logically) exist. So there is no such barber: paradox solved.

> **'The point of philosophy is to start with something so simple as not to seem worth stating, and to end with something so paradoxical that no one will believe it.'**
>
> **Bertrand Russell, 1918**

The significance of the barber paradox in fact lies not in its content but in its form. Structurally, this paradox is similar to another, more important problem known as Russell's paradox, which concerns not clean-shaven villagers but mathematical sets and their contents. This related paradox has proved a great deal less easy to solve; indeed, it is no exaggeration to say that, a century ago, it was largely responsible for undermining the very foundations of mathematics.

timeline

c.350BC Forms of argument

c.300BC The sorites paradox

'This sentence is false'

The problem of self-reference that lies at the heart of the barber paradox and Russell's paradox is shared by a number of other well-known philosophical puzzles. Perhaps most famous of all is the so-called 'liar paradox', the supposed origins of which go back to the 7th century BC, when the Greek Epimenides – a Cretan himself – is alleged to have said 'All Cretans are liars'. The simplest version is the sentence 'This sentence is false', which if true is false and if false is true. The paradox can be captured in a pair of sentences: on one side of a piece of paper – 'The sentence on the other side is false'; on the other – 'The sentence on the other side is true'. In this formulation each sentence on its own is apparently unexceptionable, so it is hard to dismiss the paradox as simply meaningless, as some have suggested.

Another interesting variant is Grelling's paradox. This involves the notion of autological words (words that describe themselves), e.g. 'pentasyllabic', which itself has five syllables; and heterological words (words that do not describe themselves), e.g. 'long', which is itself short. Every word must be of one kind or the other, so now consider: is the word 'heterological' itself heterological? If it is, it is not; if it is not, it is. There's no escape from the barber's shop, it seems.

Russell and set theory The idea of sets is fundamental to mathematics, because they are the purest objects under its scrutiny. The mathematical method involves defining groups (sets) of elements that satisfy certain criteria, such as the set of all real numbers greater than 1 or

AD1901	1905
The barber paradox	The king of France is bald

clarity of thought

Philosophical arguments are often complex and have to be expressed with great precision. Sometimes, philosophers get a little carried away by the majesty of their own intellect and trying to follow their arguments can feel like wading through treacle. If you thought the rules of cricket were hard to follow, see if you can keep up with Bertrand Russell's reasoning as he defines 'the number of a class'.

'This method is, to define as the number of a class the class of all classes similar to the given class. Membership of this class of classes (considered as a predicate) is a common property of all the similar classes and of no others; moreover every class of the set of similar classes has to the set a relation which it has to nothing else, and which every class has to its own set. Thus the conditions are completely fulfilled by this class of classes, and it has the merit of being determinate when a class is given, and of being different for two classes which are not similar. This, then, is an irreproachable definition of the number of a class in purely logical terms.'

the set of prime numbers; operations are then performed so that further properties can be deduced about the elements contained within the set or sets concerned. From a philosophical perspective, sets have been of particular interest because the recognition that all of mathematics (numbers, relations, functions) could be exhaustively formulated within set theory fuelled the ambition of using sets to ground mathematics on purely logical foundations.

At the beginning of the 20th century the German mathematician Gottlob Frege was attempting to define the whole of arithmetic in logical terms by means of set theory. At this time it was assumed that there were no restrictions on the conditions that could be used to define sets. The problem, recognized by the British philosopher Bertrand Russell in 1901,

"A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. In this position I was put by a letter from Mr Bertrand Russell as the work was nearly through the press."

Gottlob Frege, 1903

centred on the question of self-membership of sets. Some sets have themselves as members: for instance, the set of mathematical objects is itself a mathematical object. Others do not: the set of prime numbers is not itself a prime number. Now consider the set of all sets that are not members of themselves. Is this set a member of itself? If it is, it isn't; and if it isn't, it is. In other words, membership of this set depends on not being a member of the set. A straight contradiction, and hence the (barber-like) paradox. However, in contrast to the barber case, it is not possible simply to jettison the offending set – not, at any rate, without blowing a hole in set theory as it was then understood.

The existence of contradictions at the heart of set theory, exposed by Russell's paradox, showed that the mathematical definition and treatment of sets was fundamentally flawed. Given that any statement can (logically) be proved on the basis of a contradiction, it followed – disastrously – that any and every proof, while not necessarily invalid, could not be known to be valid. Mathematics basically had to be rebuilt from the foundations. The key to the solution lay in the introduction of appropriate restrictions on the principles governing set membership. Russell not only exposed the problem but was one of the first to attempt a solution, and while his own attempt was only partially successful, he helped to set others on the right path.

the condensed idea

**If it is, it isn't
if it isn't, it is**