6.2 Truth Functions

<u>1. Finding the "Main Operator":</u> Before moving on, we must learn how to identify the "main operator" within a formula. Here are some tips for finding the "**main operator**":

1) If there ARE NO parentheses, then the main operator will be the ONLY operator—unless there is more than one operator, in which case the main operator is the operator that is the one that is not a "~".

2) If there ARE parentheses or brackets, then the main operator will be the ONLY operator which is outside of all of the parenthesis or brackets—unless there is more than one operator outside of the parenthesis/brackets, in which case the main operator is the one that is not a "~".

Here are some examples.

Negation: In the following formulas, the main operator is the "~"

~A
~(A ● B)
~[(A ● B) ∨ (C ● D)]

Conjunction: In the following formulas, the main operator is the "●"

A ● B
~A ● ~B
(A ∨ B) ● (C ∨ D)
~A ● [(B ≡ C) ∨ (D ≡ E)]

Disjunction: In the following formulas, the main operator is the "∨"

A ∨ B
~A ∨ ~B
(A ● B) ∨ (C ∨ D)
~A ∨ [(B ⊃ C) ● (D ⊃ E)]

Conditional: In the following formulas, the main operator is the "⊃"

A ⊃ B
~A ⊃ B
(A ∨ B) ⊃ ~(C ∨ D)
[(A ≡ B) ● (C ≡ D)] ⊃ E

Bi-Conditional: In the following formulas, the main operator is the "≡"

A ≡ B
A ≡ ~B
~(A ● B) ≡ (B ● C)
[A ∨ (B ⊃ C)] ≡ [D ∨ (E ⊃ F)]

Whenever we find the main operator, we can reduce the entire formula to a simple formula with two letters ("p" and "q") and one single operator. For instance,

ALL of the negations above can be re-written as "~p"
ALL of the conjunctions above can be re-written as "p ● q"
ALL of the disjunctions above can be re-written as "p ∨ q"
ALL of the conditionals above can be re-written as "p ⊃ q"
ALL of the bi-conditionals above can be re-written as "p ≡ q"

For instance, if we take the very last equation under the bi-conditionals above, [A ∨ (B ⊃ C)] ≡ [D ∨ (E ⊃ F)], we may let "p" = "[A ∨ (B ⊃ C)]" and let "q" = "[D ∨ (E ⊃ F)]". In that case, we get "p ≡ q".

<u>2. Truth Functions:</u> Logicians DEFINE each of the operators in terms of their relation to the truth or falsehood of the statement they are operating on.

A. Negation: Let's consider negation first. If we have "~p", what happens when "p" is TRUE? What happens when "p" is FALSE?

To answer these questions, let's take an example:

"**It is not the case that** it is raining." Translation: ~p

Here, "p" = "it is raining". First, imagine that it IS raining (i.e., "it is raining" is true). In that case, "It is not the case that it is raining" would be a LIE. It would be FALSE. Now, imagine that it is NOT raining (i.e., "it is raining" is false). In that case, "It is not the case that it is raining would be telling the TRUTH. It would be TRUE.

In short, when "p" is true, "~p" is false. But, when "p" is false, "~p" is true. Basically, negation just takes the truth value of the thing being negated and makes it the OPPOSITE of whatever it is. For instance, since "The sky is blue" is TRUE, then "It is NOT the case that the sky is blue" is false.

This relation of the operator ("~") to the truth of the propositions being operated on ("p") is called the **truth function** of that operator. And it can be expressed in a **truth table**, like this one ("T" means "True" and "F" means "False"):

Negation

p	~p
T	F
F	T

Basically, this says that, when "p" is true, "~p" is false. And when "p" is false, "~p" is true.

Examples:

"**It is not the case that** the moon is made of cheese" is **true** because "the moon is made of cheese" is **false**.

"**It is not the case that** ocean water is salty" is **false** because "ocean water is salty" is **true**.

B. Conjunction: All of the other operators require TWO statements, "p" AND "q". What is the relation between "●" and the truth or falsehood of "p" and "q"? Well, it turns out that "p ● q" is ONLY true when BOTH "p" AND "q" are true. It will not do to have only one or the other be true. They must BOTH be true. Consider some examples:

p and q are true:
"Denver is in Colorado **and** Seattle is in Washington." (this sentence is true)

p is true, but q is false:
"The sky is blue **and** grass is red." (this sentence is false)

p is false, but q is true:
"Dirt is purple **and** the sky is blue." (this sentence is false)

p and q are both false:
"Dirt is purple **and** grass is red." (this sentence is false)

We can write the truth table for "●" as follows:

Conjunction

p	q	p ● q
T	T	T
T	F	F
F	T	F
F	F	F

C. Disjunction: What is the relation between the "∨" operator and the truth or falsehood of "p" and "q"? Well, it turns out "p ∨ q" is ONLY false when BOTH "p" AND "q" are false.

Consider some examples:

p and q are true:
"Denver is in Colorado **or** Seattle is in Washington." (this sentence is true)

Note: If the above seems counter-intuitive, remember that "or" is INCLUSIVE.

p is true, but q is false:
"The sky is blue **or** grass is red." (this sentence is true because the sky is blue)

p is false, but q is true:
"Dirt is purple **or** the sky is blue." (this sentence is true because the sky is blue)

p and q are both false:
"Dirt is purple **or** grass is red." (this sentence is false)

We can write the truth table for "∨" as follows:

Disjunction

p	**Q**	**p ∨ q**
T	T	T
T	F	T
F	T	T
F	F	F

D. Conditional: What is the relation between the "Ↄ" operator and the truth or falsehood of "p" and "q"? Well, it turns out that "p Ↄ q" is ONLY false when "p" (the antecedent) is true, and "q" (the consequent) is false.

Note: Students sometimes find this counter-intuitive, because it seems like "p Ↄ q" should turn out to be FALSE when both "p" and "q" are false. For instance, "If the moon is made of cheese, then unicorns are invading the capitol" seems totally false, but it turns out to be TRUE according to the definition of "Ↄ" that I have just given.

To understand why "Ↄ" is defined this way, it may help to consider a scenario, and ask yourself: **In which of these four scenarios did I tell a lie**?

I say to you, "If you come over and help me move my couch on Saturday, then I will buy you some pizza."

Scenario 1: You DO help me, and I DO buy you pizza (p and q are both true).
Scenario 2: You DO help me, but I do NOT buy you pizza (p is true, q is false).
Scenario 3: You do NOT help me, but I DO buy you pizza anyway (p is false, q is true).
Scenario 4: You do NOT help me, and I do NOT buy you pizza (p and q are both false).

Now, in which of these four scenarios did I tell a lie, or break my promise to you? It seems that I ONLY told a lie in the scenario where you DID come over to help me, but I did NOT buy you pizza. So, "p Ɔ q" is only false when "p" is true and "q" is false.

Alternatively, think of it this way: **Which of the 4 people are breaking this law?**

"If someone is consuming alcohol, then they are at least 21 years of age"

Scenario 1: Peggy IS consuming alcohol, and IS over 21 (p and q are both true).
Scenario 2: Sue IS consuming alcohol, but is NOT over 21 (p is true, q is false).
Scenario 3: Buddy is NOT consuming alcohol, but IS over 21 (p is false, q is true).
Scenario 4: Holly is NOT consuming alcohol, and is NOT over 21 (p and q are both false).

ONLY SUE is breaking the law. Again, "P Ɔ Q" is only violated when P is true and Q is false. We can write the truth table for "Ɔ" as follows:

Conditional

P	**q**	**p Ɔ q**
T	T	T
T	F	F
F	T	T
F	F	T

E. Bi-Conditional: Remember that "p ≡ q" is just shorthand for "(p Ɔ q) ● (q Ɔ p)". The main operator in this formula is the "●". But, "p ● q" only comes out true when BOTH "p" AND "q" are true. So, likewise, "(p Ɔ q) ● (q Ɔ p)" only comes out true when BOTH "(p Ɔ q)" AND "(q Ɔ p)" are true.

Imagine in the story above that I told you, "If you come over and help me move my couch, then I will buy you pizza" AND YOU also added, "Sure Chad. If you buy me pizza, then I will come over and help you." NOW consider scenarios 1 through 4 above.
In which of the four scenarios has one of us told a lie? Now scenarios 2 AND 3 involve telling lies. In scenario 2, you help me, but I don't buy you pizza. In scenario 3, I buy you pizza, but you don't come over and help me.

In short, for “p ≡ q” to be true, “p” and “q” must either BOTH be false, or else they must BOTH be true. They must stand or fall together. We can write the truth table for “≡” as follows:

Bi-Conditional

P	**q**	**p ≡ q**
T	T	T
T	F	F
F	T	F
F	F	T

3. Calculating the Truth of Propositions: Now, let’s use what we’ve learned to determine the truth value of some longer propositions.

If I give you something simple, such as “A ● B” and tell you that “A” is true and “B” is false, then what is the truth value of “A ● B”? It is false. Have another look at the truth table for “●” and see that “p ● q” ONLY comes out true if BOTH of the conjuncts are true.

Example #1: Let’s try a harder one:

> “If Either Obama or Mitt Romney wins the election, then the world will end in 2012.”

In symbolic form, this becomes:

(O ∨ M) Ↄ W   O=true, M=false, W=false

Using “T” for “True” and “F” for “False”, we can re-write this as follows:

(T ∨ F) Ↄ F

The main operator in the formula above is the “Ↄ”. We get rid of one operator at a time, and save the main operator for last. So, first let’s focus on the “T ∨ F” portion of the formula. What is the truth value of “p ∨ q” when “p” is true and “q” is false? It’s true! So, we know that “T ∨ F” is true. We can replace the entire disjunction with a “T”, like this:

T Ↄ F

Now, what is the truth value of “p Ↄ q” when “p” is true” and “q” is false? It’s false! So, the WHOLE sentence that we originally started with is false. In other words, “If either Obama or Mitt Romney wins the election, then the world will end in 2012” is FALSE.

Example #2: Here is another example:

~[(A ● B) ∨ (C ● D)] A=true, B=true, C=true, D=false

Is this sentence true or false? Using our truth tables above, we get the following steps:

~[(T ● T) ∨ (T ● F)]	(we replace the letters with "T" or "F")
~(T ∨ F)	(the first conjunction is true; the second one is false)
~ T	(the disjunction is true)
F	(the negation of a truth is a falsehood)

Answer: False

Example #3: Let's try one more example:

[A ∨ (B Ɔ C)] ≡ [D ∨ (E Ɔ F)] A=false, B=true, C=false, D=false, E=true, F=false

Is this sentence true or false? Using our truth tables above, we get the following steps:

[F ∨ (T Ɔ F)] ≡ [F ∨ (T Ɔ F)]	(we replace the letters with "T" or "F")
(F ∨ F) ≡ (F ∨ F)	(both of the conditionals in the parentheses are false)
F ≡ F	(both of the disjunctions are false)
T	(a bi-conditional is TRUE when both sides are false)

Answer: True

Note: Do homework for section 6.2 at this time.